Finite Element Computational Homogenization of Nonlinear Multiscale Materials in Magnetostatics

I. Niyonzima [1], R. V. Sabariego [1], P. Dular [1,2] and C. Geuzaine [1]

[1] Dept. of Electrical Engineering and Computer Science, University of Liège, Belgium, iniyonzima@doct.ulg.ac.be
[2] Fonds de la Recherche Scientifique - FNRS, Belgium

***Abstract*—This paper deals with the modelling of nonlinear multiscale materials in magnetostatics by means of a finite element computational homogenization method. The method couples a macroscale problem with many microscale problems. During the upscaling step, the homogenized magnetic permeability and its derivative with respect to the magnetic field are calculated from the microscale solution and transferred to the macroscale. The downscaling step consists in imposing proper boundary conditions for the microscale problems from the macroscale solution. Results are validated by comparison with those obtained with classical finite element brute force approach.**

I. Introduction

Over the last few years, several multiscale computational methods have been proposed to study multiscale materials, mainly in the frame of mechanical, fluid dynamic and thermal problems. Among many others, it is worth mentioning the Multiscale Finite Element Methods (MsFEM) [1] and the Heterogeneous Multiscale Methods (HMM) [2], [3]. The former construct adapted global basis functions for the macroscale problem by solving microscale problems. The latter solve the microscale problems for determining a homogenized or average quantity of interest that is directly transfered to the macroscale problem. Both approaches take advantage of the separation of scales, with possibly different governing equations for the considered scales. However, while the HMM yields to a greatly reduced computational cost, MsFE methods are often as expensive as a brute force technique. A very popular HMM-type method is the so-called FE^2 method that applies FE to solve the micro and macro problems [4].

In this paper, we apply a FE computational method within the HMM framework to a nonlinear multiscale magnetostatic problem. The method couples problems at two different scales:

- the macroscale problem that accounts for the slowly varying component of the full solution;
- microscale problems that fully resolve the material inhomogeneities at the smallest scale.

The macroscale solution serves to impose suitable boundary conditions for the microscale problems. In turn, the solution of these microscale problems allows to calculate the effective magnetic permeability and its derivative with respect to the magnetic field (for Newton-Raphson iterations) for the macroscale problem.

II. Magnetostatic problem

The magnetostatic problem in a bounded domain $\Omega = \Omega_s \cup \Omega_s^C \in \mathbb{R}^3$ is defined by the following Maxwell equations and constitutive law [5]:

$$\operatorname{curl} \underline{h}(\boldsymbol{x}) = \underline{j}(\boldsymbol{x})\,, \quad \operatorname{div} \underline{b}(\boldsymbol{x}) = 0\,, \quad \underline{b}(\boldsymbol{x}) = \mu(\boldsymbol{x}, \underline{h}(\boldsymbol{x}))\underline{h}(\boldsymbol{x})\,, \tag{1 a-c}$$

with $\boldsymbol{x}$ the space position, $\underline{h}$ the magnetic field, $\underline{b}$ the magnetic flux density, $\underline{j}$ the electric current density and μ the permeability. Domain Ω_s contains the sources and Ω_s^C denotes its complement. Proper boundary conditions must also be imposed.

We use the scalar potential formulation and decompose $\underline{h}$ into a source term and a reaction term. Assuming that the domain is simply connected, $\underline{h}(\boldsymbol{x}) = \underline{h}_s(\boldsymbol{x}) - \operatorname{grad}\phi(\boldsymbol{x})$ where ϕ is the magnetic scalar potential. Then, the weak form of (1 b) leads to [5]: Find $\phi(\boldsymbol{x})$ such that

$$\Big(\mu(\boldsymbol{x}, \underline{h}(\boldsymbol{x}))\cdot\operatorname{grad}\phi(\boldsymbol{x}), \operatorname{grad}\phi'(\boldsymbol{x})\Big)_\Omega = \Big(\underline{h}_s(\boldsymbol{x}), \operatorname{grad}\phi'(\boldsymbol{x})\Big)_\Omega \tag{2}$$

holds for all test functions $\phi'(\boldsymbol{x})$ in an appropriate function space.

III. Computational homogenization model

In a multiscale material, rapid spatial variations of the magnetic permeability induce rapid variations of the magnetic scalar potential $\phi^\varepsilon(\boldsymbol{x})$. The exponent ε refers to the ratio between the scale of the material and the scale of its microstructures, hence it denotes quantities with rapid spatial variations. Let us assume $\underline{h}_s(\boldsymbol{x}) = \underline{0}$ and impose $\phi^\varepsilon(\boldsymbol{x})$ on some parts of the boundary, the magnetic field becomes $\underline{h}^\varepsilon(\boldsymbol{x}) = -\operatorname{grad}\phi^\varepsilon(\boldsymbol{x})$ and the weak form (2) reads: Find $\phi^\varepsilon(\boldsymbol{x})$ such that

$$\Big(\mu^\varepsilon(\boldsymbol{x}, \underline{h}^\varepsilon(\boldsymbol{x}))\cdot\operatorname{grad}\phi^\varepsilon(\boldsymbol{x}), \operatorname{grad}{\phi'}^\varepsilon(\boldsymbol{x})\Big)_\Omega = 0 \tag{3}$$

is verified for all ${\phi'}^\varepsilon(\boldsymbol{x})$ in a suitable basis function space.

Equation (3) can be solved in the whole domain using e.g. finite element method. However this is very expensive in terms of memory and computation time due to the need of discretizing the unknown field at the smallest scale ε. Finite element computational homogenization methods allow to overcome this problem. The principle of the method is explained in Fig. 1. A macroscale problem is defined on a coarse mesh covering the entire domain and many microscale problems are defined on small, finely meshed areas around some points of interest of the macroscale mesh (e.g. numerical quadrature points). In the following, the subscripts M and m refer to macroscale and microscale quantities, respectively.

A. Downscaling

From (3), the weak equation to solve at the microscopic level reads:

$$\Big(\mu^\varepsilon(\boldsymbol{x}, \underline{h}^\varepsilon_m(\boldsymbol{x}))\cdot\operatorname{grad}\phi^\varepsilon_m(\boldsymbol{x}), \operatorname{grad}{\phi'}^\varepsilon_m(\boldsymbol{x})\Big)_{\Omega_m} = 0\,, \tag{4}$$

Work supported by the Belgian French Community (ARC 09/14-02) and the Belgian Science Policy (IAP P6/21). I. Niyonzima is a fellowship beneficiary with the Belgian Research Training Fund for Industry and Agriculture (FRIA).

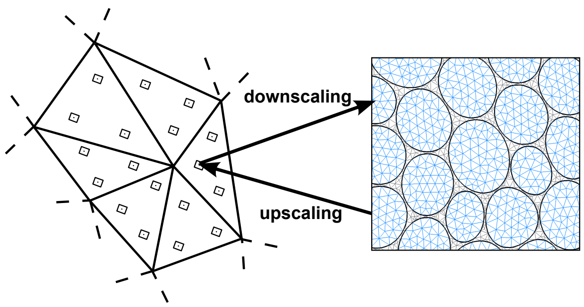

Fig. 1. Scale transitions between macroscale (left) and microscale (right) problems. Downscaling (Macro to micro): obtaining proper boundary conditions for the microscale problem from the macroscale solution. Upscaling (micro to Macro): effective quantities for the macroscale problem calculated from the microscale solution

where Ω_m is the microdomain. This equation must be completed by the boundary conditions obtained from the macroscale solution as explained hereafter.

The microscale magnetic scalar potential ϕ_m^ε can be expressed in terms of ϕ_M, the mean macroscale component with slow variations, and ϕ_c^ε, a correction term that accounts for the rapid variations, i.e.

$$\phi_m^\varepsilon(\boldsymbol{x}) = \phi_M(\boldsymbol{x}) + \phi_c^\varepsilon(\boldsymbol{x}) \,. \quad (5)$$

Applying the gradient operator to both sides of (5) and integrating gives:

$$\frac{1}{V_m}\int_{\Omega_m} \operatorname{grad} \phi_m^\varepsilon(\boldsymbol{x})\,\mathrm{d}\Omega_m = \frac{1}{V_m}\int_{\Omega_m} \operatorname{grad} \phi_M(\boldsymbol{x})\,\mathrm{d}\Omega_m + \frac{1}{V_m}\int_{\Gamma_m} \boldsymbol{n}\phi_c^\varepsilon(\boldsymbol{x})\,\mathrm{d}\Gamma_m \quad (6)$$

where V_m and Γ_m are respectively the volume and the boundary of the microdomain Ω_m. Assuming that the average magnetic field is conserved, we can write:

$$\frac{1}{V_m}\int_{\Omega_m} \operatorname{grad} \phi_m^\varepsilon(\boldsymbol{x})\,\mathrm{d}\Omega_m = \frac{1}{V_m}\int_{\Omega_m} \operatorname{grad} \phi_M(\boldsymbol{x})\mathrm{d}\Omega_m \quad (7)$$

which implies that the magnetic field is consistent between the macroscale and the microscale. Furthermore, it infers periodic boundary conditions for the correction term ϕ_c^ε. Note that the surface integral in (6) vanishes.

B. Upscaling

If scale separation holds, the following nonlinear equation governs the macroscale problem:

$$\Big(\mu_M(\boldsymbol{x}, \underline{h}_M(\boldsymbol{x})) \cdot \operatorname{grad} \phi_M(\boldsymbol{x}), \operatorname{grad} \phi'_M(\boldsymbol{x})\Big)_{\Omega_M} = 0 \,. \quad (8)$$

The effective magnetic permeability μ_M is calculated by equalizing the magnetic co-energies at the microscale and the macroscale:

$$\frac{1}{V_m}\int_{\Omega_m} \underline{h}_m^\varepsilon(\boldsymbol{x}) \cdot \mu^\varepsilon\Big(\boldsymbol{x}, \underline{h}_m^\varepsilon(\boldsymbol{x})\Big) \cdot \underline{h}_m^\varepsilon(\boldsymbol{x})\mathrm{d}\Omega_m = \underline{h}_M(\boldsymbol{x}) \cdot \mu_M\Big(\boldsymbol{x}, \underline{h}_M(\boldsymbol{x})\Big) \cdot \underline{h}_M(\boldsymbol{x}) \,. \quad (9)$$

The left side of (9) can be calculated from the microscale solution and $\underline{h}_M$ is deduced from (7). We use the finite element method to solve both the macroscale problem and the microscale problems.

IV. Applications

In order to validate the new computational homogenization method we solve a one-dimensional problem and compare the results to those obtained by the classical finite element method with an extremely fine mesh. We adopt the following rapidly varying permeability:

$$\mu^\varepsilon\Big(\boldsymbol{x}, \underline{h}_m^\varepsilon(\boldsymbol{x})\Big) = \frac{\mu_0\Big[1 + \cos\Big(2\pi\frac{x}{\varepsilon}\Big)\Big]}{|\underline{h}_m^\varepsilon(\boldsymbol{x})| + 0.5\exp\Big(\frac{0.5 - |\underline{h}_m^\varepsilon(\boldsymbol{x})|}{5}\Big)} \,. \quad (10)$$

A zoom of the magnetic field on the microdomain defined on the interval $[0.225, 0.375]$ is depicted in Fig. 2.

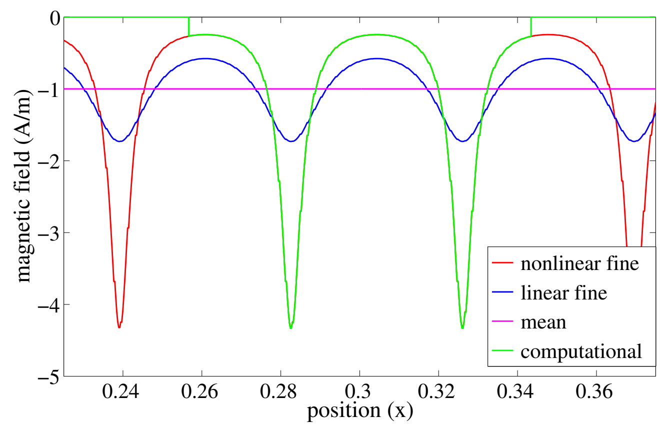

Fig. 2. Comparison of magnetic field obtained using different assumptions

The curves labeled “fine” correspond to the reference nonlinear and linear solutions, i.e. the macro problem is solved on a very fine mesh that accounts for the variations of μ^ε in (10). The result of the proposed “computational” approach is obtained when coupling the macro problem on a very coarse mesh and micro problems on a fine mesh (one per e.g. each integration point at the macro level) on a fine mesh. The local magnetic field is shown to be very accurate at the microscale.

In the extended paper, we will apply the method to a nonlinear magnetostatic problem, the lamination stacks that can be found in transformers or electric machines. A discussion on the computational cost will be included as well.

References

[1] Y. R. Efendiev, T. Hou, and V. Ginting, “Multiscale finite element methods for nonlinear problems and their applications,” *Comm. Math. Sci.*, vol. 2, no. 4, pp. 553–589, 2004.
[2] W. E and B. Engquist, “The heterogeneous multiscale methods,” *Comm. Math. Sci.*, vol. 1, no. 1, pp. 87–132, 2003.
[3] W. E, P.-B. Ming, and P.-W. Zhang, “Analysis of the Heterogeneous Multiscale Method for Elliptic Homogenization Problems,” *Journal of the American Mathematical Society*, vol. 18, no. 1, pp. 121–156, 2004.
[4] I. Ozdemir, M. Brekelmans, and M. Geers, “FE2 computational homogenization for the thermo-mechanical analysis of heterogeneous solids,” *Comput. Methods Appl. Mech. Engrg.*, vol. 198, pp. 602–613, 2008.
[5] A. Bossavit, *Computational Electromagnetism. Variational Formulations, Complementarity, Edge Elements*. Academic Press, 1998.